A Detailed List and a Periodic Table of Set Classes

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Abstract

In this paper, pitch-class sets are analyzed in terms of their intervallic structures and those related by transposition are called a set type. Then, non-inversionally-symmetrical set classes are split into two set types related by inversion. As a higher version of the interval-class vector, I introduce the trichord-type vector, whose elements are the number of times each trichord type is contained in a set type, as well as a trichord-class vector for set classes. By using the interval-class, trichord-class, and trichord-type vectors, a list of set classes and types is developed, including, apart from the usual information, the intervallic structures and the trichord-type vectors. The inclusion of this last characteristic is the most significant difference with respect to previously published lists of set classes. Finally, a compact periodic table containing all set classes is given, showing their main characteristics and relationships at a glance.

Keywords: list of set classes, interval-class vector, trichord-class vector, trichord-type vector, periodic table

1. Introduction

Pitch-class set theory or, in a broad sense, post-tonal theory, has been consolidated during the second half of the twentieth century and has proved to be a powerful tool for composing and analyzing atonal music, and can be applied to tonal music as well. Babbitt (1955, 1960, 1961), Lewin (1959, 1960), Martino (1961) and Perle (1991 [1962]) developed the main concepts and applied them to twelve-tone composition. Forte (1973) introduced his set-class names and considered the pitch-class sets in a general context, not necessarily in twelve-tone music. Rahn (1980) rewrote much of this theory in a more formal and mathematical style and provided additional results. Further mathematical developments were given by Lewin (2011 [1987]) and Morris (1987, 2001). Additionally, an overview of mathematics involved in the development of twelve-tone system is given by Morris (2007).

As a brief summary, pitch-class sets related by transposition or inversion are grouped into a set class. We represent every set class by a simple form, give it a name, and determine its symmetries and complement. Additionally, the interval-class vector lists the interval classes contained in each set class, which characterizes, to a great extent, its sonority (but not completely!). Lists of set classes including all that information are an essential part of this theory and are available in many texts.

In this paper, the basic concepts are introduced through group theory and, particularly, group actions. This way, pitch classes are considered as elements of the set of integers modulo 12, and all pitch-class sets related by transposition, here called a set type, are defined as the orbit of a pitch-class set under the action of the group of transpositions. Then, I represent a set type by the so-called intervallic form or IF (the intervals between every two adjacent pitch classes, including the interval between the last and the first ones), so that we can easily derive its inversion and its complement. Furthermore, each set class not being inversionally symmetrical is split into two set types related by inversion, which allows distinguishing, for example, between major and minor triads, or between dominant and half-diminished seventh chords.

As a higher version of the interval-class vector (ICV), I introduce the so-called trichord-type vector (TTV), whose elements are the number of times each trichord type is contained in a set type. I find that the TTV fully characterizes the sonority of a set type, except in only one case. As well, when dealing with set classes instead of set types, I obtain a reduced version of the TTV, which is here called the trichord-class vector (TCV). It is easily obtained from the TTV and, in contrast to the ICV, fully characterizes a set class.

Then, I provide a detailed list of set classes and types, which adds the IF and the TTV to similar lists published in Forte (1973), Rahn (1980) or Straus (2016). Finally, I develop a compact periodic table including all set classes, which shows their main characteristics and relationships at a glance.

There are many texts and articles that provide full explanation of the basic concepts and terminology used in this paper, Straus (1991) being a primer and Straus (2016) an undergraduate textbook. However, there are sometimes slight differences among the terms and notations used by different authors. For this reason, sections 2 and 3 give a summary of those concepts and terms, along with the notations and acronyms here used. Section 2 also gives the definition of the IF, together with its relevant properties.

Section 4 deals with the TTV, including the formulas relating the TTVs of a set type and its complement, although they are derived in Appendix 1. The TCV is also introduced here. Section 5 describes the detailed list of set classes and types, which is given in Appendix 2. And section 6 includes the above-mentioned periodic table. The ICV, TCV and TTV played a crucial role in arranging the set classes and types, both in the detailed list and the periodic table.

2. Intervallic Form

The 12 pitch classes are represented as elements of the set of integers modulo 12, $\mathbb{Z}_{12}$, where 0 corresponds to note C and 11 to note B. In this study, integers 10, 11 and 12 are represented by letters A, B and C, respectively (C is only used in one line in Appendix 2). A pitch-class set is then a subset of $\mathbb{Z}_{12}$ and is written in brackets without commas; for example, [95A24].

Following Crans, Fiore, & Satyendra (2009), I consider the cyclic group of transpositions T of order 12 and the dihedral group of transpositions and inversions T/I of order 24. Then, a *set type* is here defined as the orbit of a pitch-class set under the action of the cyclic group T, whereas a *set class* is the orbit of a pitch-class set under the action of the dihedral group T/I. Since the inversion of a pitch-class set also gives rise to a set type, a set class is formed by a set type and its inversion.

For a pitch-class set, its *degree of transpositional symmetry* s_T is the order of its stabilizer in the cyclic group T; and, similarly, we can define its *degree of dihedral symmetry* $s_{T/I}$ as the order of its stabilizer in the dihedral group T/I. Therefore, from the orbit-stabilizer theorem, the number of different pitch-class sets in a set type is $12/s_T$ and the number of those in a set class, $24/s_{T/I}$. In some references (as Straus 2016), the *degree of inversional symmetry* s_I is also defined, which is related to the previous ones by $s_I = s_{T/I} - s_T$ (or $s_{T/I} = s_T + s_I$). The value of s_T must be a divisor of 12 and $s_{T/I}$ is equal either to s_T ($s_I = 0$) or $2s_T$ ($s_I = s_T$), in which case the pitch-class set is said to be *inversionally symmetrical*. All pitch-class sets in a set class, as well as their complements, have the same values of s_T and $s_{T/I}$ (and, therefore, s_I).

For simplicity, it is common to write pitch-class sets and set types in their *normal forms*. There are, however, two widely used normal forms, one given by Forte (1973) and the other by Rahn (1980), and they are not always equal. In each case, the lesser of the normal forms of a set type and its inversion, with respect to the lexicographic order, is the corresponding *prime form*. Unless otherwise indicated, the lexicographic order will be assumed when comparing normal forms or vectors (see below).

In this study, another normal form, together with its corresponding prime form, are considered. Given a pitch-class set, such as [95A24], the pitch classes are first written in ascending order within an octave, for example [59A24], and then the differences from each integer to the next one are obtained, which gives {41421} (the last integer being the difference from the last to the first integer in the pitch-class set). This result, or any of its circular shifts, is here called the *intervallic form* (IF) and will be written in braces. It represents the set type of the given pitch-class set. Since an IF contains the sequence of ordered pitch-class intervals in a pitch-class set, the sum of all integers in any IF is always 12. The least of the circular shifts of an IF (with respect to the lexicographic order) will be the *normal intervallic form*. In this example, it is {14142}. The IF is the same as the "interval notation" given by Regener (1974), except that he chooses as normal form the greatest in lexicographic order and does not consider a prime form, which may cause difficulties for relating the two set types of a set class when arranging them. The IF has several important properties:

1) Given a set type, the IF easily allows obtaining both the Forte and Rahn normal forms. The latter corresponds to the circular shift of the IF which is the greatest in colexicographic order. In this example, it is {21414}, thus giving the pitch-class set starting from 0: [02378], which is actually the Rahn normal form. With respect to the Forte normal form, it corresponds to the circular shift of the IF which has the maximum integer on the right and, if there are several options (as in this example, where there are two 4's), the least in lexicographic order. In this example, it is {14214}, thus giving the pitch-class set starting from 0: [01578], which is actually the Forte normal form. Although in this example the Forte and Rahn normal forms are different, in most cases they are the same.

2) Contrary to Forte and Rahn normal forms, the IF of a pitch-class set allows obtaining the IF of its inversion directly: it is just the same IF but in reverse order. So, in this example, it is {12414}, which is also the normal IF. Therefore, its corresponding Rahn normal form is obtained from the circular shift {14124} (the greatest in colexicographic order), which gives the pitch-class set starting from 0: [01568]. And its Forte normal form is obtained directly from {12414} (because it already has the maximum integer on the right and is the least circular shift in lexicographic order), which gives the pitch-class set starting from 0: [01378]. Given the normal IFs of a set type and its inversion, the lesser will be the *prime intervallic form* of the corresponding set class. In this example, it is {12414}. Normally, as in this example, it also corresponds to the Forte and Rahn prime forms, but this is not always true.

3) Contrary to Forte and Rahn normal forms, the IF of a pitch-class set easily allows obtaining the IF of its complement, too. For example, the IF of [59A24] is {41421}, which can be mentally represented as in Figure 1-a). Its complement is shown in Figure 1-b), whose IF will be {1131123} or any of its circular shifts.

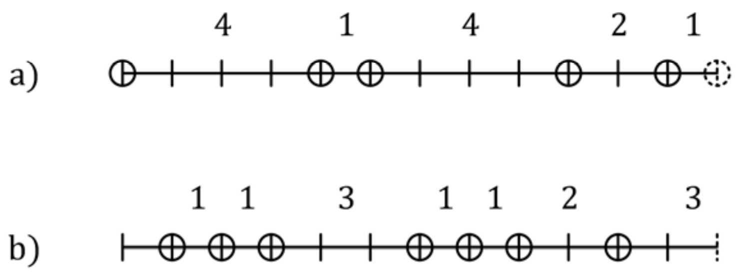

Figure 1. IFs of a pitch-class set and its complement.

To sum up, for a set type, it is simple to obtain the IF, as well as the IF of its inversion, its complement and, consequently, the inversion of its complement (which is equal to the complement of its inversion). And these IFs easily allow obtaining the corresponding Forte and Rahn normal forms. On the contrary, obtaining all these normal forms from one of them, either Forte or Rahn, without using the IF, is laborious. Additionally, the IF easily allows determining whether or not a set class is inversionally symmetrical (by comparing the IF in both directions), as well as obtaining its degree of transpositional symmetry (which is equal to the number of periods in the IF, analyzed as a periodic structure). In short, the IF is a simple and versatile representation of set types.

3. Interval-Class Vector

The number of pitch classes in a pitch-class set or a set class is its cardinality and will be represented by c. For $c = 2$, there are 6 different set classes: the dyads, also called the unordered pitch-class intervals or interval classes, which will be arranged by increasing prime IF. All of them are inversionally symmetrical and have $s_I = s_T = 1$, except {66}, the tritone, which has $s_I = s_T = 2$.

For $c > 2$, the interval-class vector (ICV) lists the number of times each of the 6 dyads is contained in a given set class. This characterizes, to a great extent, the sonority of the set class, but not completely. For a set class with cardinality c, the sum of the elements of the ICV is $\binom{c}{2} = c(c-1)/2$. The ICVs of set classes are given in many sources, as well as in Appendix 2.

The ICV is defined for the action of the dihedral group T/I, so all pitch-class sets forming a set class have the same ICV. However, there are some different set classes with the same ICV. They are said to be Z-related and no more than 2 different set classes have the same ICV (this is true in $\mathbb{Z}_{12}$, but for a study of the Z-relation in $\mathbb{Z}_n$ see Mandereau et al. 2011). On the other hand, there is a simple and well-known relationship between the ICVs of a set class and its complement, derived in Appendix 1. Given a set class with cardinality c and ICV $[d_1 \cdots d_6]$, its complement will have the cardinality $c' = 12 - c$ and ICV $[d'_1 \cdots d'_6]$, which can be obtained by

$$\begin{bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ d'_4 \\ d'_5 \\ d'_6 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} + (6 - c) \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}. \tag{1}$$

All elements in the last vector are 2 except the last one, which is 1, because the dyad $\{66\}$ has $s_T = 2$. For the rest of elements, the difference between the two ICVs is $2(6 - c)$, which equals $c' - c$. In the case of hexachords, $6 - c = 0$, so each hexachord and its complement have the same ICV. Therefore, if a hexachord class is not Z-related to any other one, then it is self-complementary. Conversely, if a hexachord class is Z-related to another one, then it is its complement (as shown at the end of section 4).

From (1), it follows that if set classes with the same cardinality are arranged with their ICVs in increasing or decreasing order, then so will the ICVs of their complements. This will make a set class and its complement to be in the same column in Table 1 (see section 6).

4. Trichord-Type and Trichord-Class Vectors

For $c = 3$, there are 12 different set classes: the trichords, which will be arranged by decreasing ICV (or increasing prime IF). All of them have $s_T = 1$, except $\{444\}$, the augmented triad, which has $s_T = 3$. Five of them are inversionally symmetrical (including $\{444\}$). Thus, each of the remaining seven trichord classes is formed by two different trichord types, related by inversion. Both have the same ICV, but their sonorities are different, as well as their normal IFs. So, the one with lesser normal IF will be called “type a” and the other “type b”. For example, the trichord class {345} represents both the minor and major triads, whose normal IFs are, respectively, {345} and {354}, the former being type a and the latter type b. The total number of trichord types is, therefore, $5 + 7 \times 2 = 19$.

For $c > 3$, we can obtain the number of times each of the 19 trichord types is contained in a given set type. The result is a 19-element vector, which is here called the *trichord-type vector* (TTV). For clarity, the TTV will be written as two groups of numbers separated by a hyphen, the first one including the first 9 elements of the vector (corresponding to trichord types {11A}, {129}, {192}, {138}, {183}, {147}, {174}, {156}, and {165}, that is, those containing semitones) and the second one including the other 10 (corresponding to trichord types {228}, {237}, {273}, {246}, {264}, {255}, {336}, {345}, {354}, and {444}, that is, those not containing semitones). For a set type with cardinality c, the sum of the elements of the TTV is $\binom{c}{3} = c(c-1)(c-2)/6$. Appendix 2 contains the TTVs of all set classes and types, obtained with an original program written in MATLAB. It has been found that only 2 of them have the same TTV, which are the two types of set class {112143}. The rest of them have a unique TTV and therefore it fully characterizes their sonorities.

Let us consider a set class with two types. If the TTV of one type (a or b) is $t_1 \cdots t_{19}$, then the TTV of the other type is obtained by simply interchanging the elements corresponding to non-inversionally-symmetrical trichords, as indicated in Figure 2. If a set class is inversionally symmetrical (only one type), the elements in the pairs shown in this figure are the same.

$$t_1 \quad \underbrace{t_2 \quad t_3}_{\curvearrowright} \quad \underbrace{t_4 \quad t_5}_{\curvearrowright} \quad \underbrace{t_6 \quad t_7}_{\curvearrowright} \quad \underbrace{t_8 \quad t_9}_{\curvearrowright} \quad - \quad t_{10} \quad \underbrace{t_{11} \quad t_{12}}_{\curvearrowright} \quad \underbrace{t_{13} \quad t_{14}}_{\curvearrowright} \quad t_{15} \quad t_{16} \quad \underbrace{t_{17} \quad t_{18}}_{\curvearrowright} \quad t_{19}$$

Figure 2. Relation between the TTVs of types a and b of a set class.

It is also possible to define a *trichord-class vector* (TCV) as the number of times each of the 12 trichord classes is contained in a given set class. The result is now a 12-element vector, whose elements are easily obtained from the TTV by simply adding the elements corresponding to non-inversionally-symmetrical trichords, as indicated in

Figure 3, where $t_1 \cdots t_{19}$ are the elements of a TTV (either type, a or b) and $f_1 \cdots f_{12}$ those of the TCV. Thus, the TTV is defined for the action of the cyclic group T, whereas the TCV is defined for the action of the dihedral group T/I. No two set classes have the same TCV, so the TCV fully characterizes a set class (this is true in $\mathbb{Z}_{12}$, but for the corresponding study in $\mathbb{Z}_n$ see the Z^3-relation in Mandereau et al. 2011).

$$\begin{array}{cccccccccccccc} t_1 & t_2 \; t_3 & t_4 \; t_5 & t_6 \; t_7 & t_8 \; t_9 & - & t_{10} & t_{11} \; t_{12} & t_{13} \; t_{14} & t_{15} & t_{16} & t_{17} \; t_{18} & t_{19} \\ \downarrow & \underbrace{+} & \underbrace{+} & \underbrace{+} & \underbrace{+} & & \downarrow & \underbrace{+} & \underbrace{+} & \downarrow & \downarrow & \underbrace{+} & \downarrow \\ f_1 & f_2 & f_3 & f_4 & f_5 & & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} \end{array}$$

Figure 3. Relation between a TTV and the TCV of a set class.

Appendix 1 derives the relationship between the TTVs of a set type and its complement. Given a set type with cardinality c, TTV $[t_1 \cdots t_{19}]$ and ICV $[d_1, \dots, d_6]$, its complement will have the TTV $[t'_1 \cdots t'_{19}]$, which can be obtained by

$$\begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ \vdots \\ t'_{18} \\ t'_{19} \end{bmatrix} = \mathrm{T_h} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix} + (4-c) \begin{bmatrix} 3 \\ 3 \\ 3 \\ \vdots \\ 3 \\ 1 \end{bmatrix}, \qquad (2)$$

where $\mathrm{T_h}$ is the matrix whose rows are the ICVs of the 19 trichord types, but multiplying the last column by 2 (the s_T of {66}) and dividing the last row by 3 (the s_T of {444}), and is given in (A13). All elements in the last vector are 3 except the last one, which is 1, because the trichord type {444} has $s_T = 3$.

On the other hand, the ICV and the TTV of a set type are related by

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \frac{1}{c-2} \mathrm{T'} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix}, \; c \geq 3, \qquad (3)$$

where T' is the matrix whose columns are the ICVs of the 19 trichord types, which is the transpose of matrix T in (A10).

The TTVs of a set type and its complement are then related by

$$\begin{bmatrix} t_1' \\ t_2' \\ t_3' \\ \vdots \\ t_{18}' \\ t_{19}' \end{bmatrix} = \left(\frac{1}{c-2}\mathrm{T_h T'} - \mathrm{I}\right)\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix} + (4-c)\begin{bmatrix} 3 \\ 3 \\ 3 \\ \vdots \\ 3 \\ 1 \end{bmatrix},\ c \geq 3, \qquad (4)$$

where I is the identity matrix of size 19, and the matrix product $\mathrm{T_h T'}$ is given in (A17).

For set classes not being inversionally symmetrical and with cardinality $c > 3$, the type with lexicographically greater TTV will be called type a, and the other type b. For example, the set class {2334} is formed by the types {2334} and {2433}, corresponding to the half-diminished and dominant seventh chords, respectively, the former having greater TTV (as can be seen in Appendix 2), thus being type a. This way, the complement of an a-type is always a b-type and vice versa. This can be seen directly from (2), where the greater $[t_1 \cdots t_{19}]$ gives the lesser $[t_1' \cdots t_{19}']$ and vice versa (the ICV $[d_1, \ldots, d_6]$ is the same for both types of a set class). The only exception is the set class {112143}, since its two types have the same TTV. Moreover, in this case, the complement of each type is itself; that is, they are self-complementary. So, in this case, the type a will be the one with lesser normal IF, that is, the prime IF. With these criteria, most prime IFs happen to be type a.

For each pair of Z-related set classes, the member with greater TCV is here called *hard* (because it turns out that it has the smaller intervals closer together) and the other *soft*. From (2), and taking into account Figure 3, it is obvious that the complement of a hard set class is a soft one and vice versa. Since this is also true for the hexachords, if a hexachord class is Z-related to another one, then it is its complement. For example, set classes {12216} and {11235} are Z-related, the latter having greater TCV, thus being hard, and the former soft. Their complements are, respectively, {1111323} (hard) and {1112124} (soft). These relations will be best visualized in Table 1, where these set classes correspond to 5-Z12, 5-Z36, 7-Z12, and 7-Z36 (see section 6).

5. List of Set Classes and Types

Usually, lists of set classes are given in one or several tables including the relevant information for each set class, such as: prime form (Forte or Rahn), Forte name, ICV and the degrees of transpositional and inversional symmetries (see, for example, Straus 2016). As well, every set class is placed across from its complement. Thus, on the one side (left) there are the trichords, tetrachords and pentachords, and on the other side (right) the nonachords, octachords, and heptachords. The hexachords form a separate group, where only those being Z-related are placed across from each other.

Forte names consist of two numbers separated by a hyphen, the first one corresponding to the cardinality and the second to an ordinal. For set classes with the same cardinality the ordinals are assigned by decreasing ICV, except in the case of Z-related pairs, where one member of each pair is placed at the end of the corresponding group. This way, every set class and its complement have the same ordinal. For example, the set class {12414} considered in section 2 is named 5-20, its complement being 7-20. Forte names of Z-related set classes include the letter “Z” just before the ordinal.

Appendix 2 is a detailed list of set classes and types. They are first grouped by cardinality and then arranged by decreasing ICV, with the ties being arranged by decreasing TCV (the hard set class in a Z-related pair before the soft one) and further ties by decreasing TTV (the a-type before the b-type). The only remaining tie (the two types of set class {112143} or 6-14) is arranged by increasing normal IF. Complementary set classes are placed “next to” each other (vertically in the case of hexachords and horizontally in the other cases). To simplify the notation, no brackets or braces are used in the list. For each set class or type, the information is given in 6 columns containing the following:

1) A general ordinal, ranging from 0 to 351. Set classes being inversionally symmetrical ($s_I = s_T$) are indicated by a hyphen just after the ordinal (no hyphen means $s_I = 0$). Regarding the degree of transpositional symmetry, when $s_T > 1$ it is given as a superscript after the ordinal or the hyphen (no superscript means $s_T = 1$).

2) The normal IF. For most non-inversionally-symmetrical set classes ($s_I = 0$), their prime IFs are type a, but when they are type b an asterisk (*) is included just after the normal IF.
3) The Rahn normal form. For most non-inversionally-symmetrical set classes, their Rahn prime forms are type a, but when they are type b it is indicated by a superscript with a plus sign (+). As well, most Forte normal forms are equal to Rahn's, but when different a superscript with a letter is included, which refers to the Forte normal form at the end of the list (none of them turn out to have the plus sign).
4) An extended Forte name, including a letter ("a" or "b") to indicate the type, when applicable. As well, Z-related set classes include a superscript with the ordinal of the other member of the pair. When looking for the complement of a set type, remember that the complement of an a-type is a b-type and vice versa, except for the two types of set class {112143} (6-14a and 6-14b), which are self-complementary. This special case is indicated by a superscript with an equal sign (=), representing that both types have the same TTV and type a corresponds to the prime IF.
5) The ICV.
6) The TTV. The inclusion of this characteristic is the most significant difference with respect to previously published lists of set classes.

6. Periodic Table of Set Classes

It would be interesting to have a compact version of the list of set classes and types, where the main information and the relationship among them can be seen at a glance. Table 1 serves this purpose. As will be explained below, it contains all set classes represented by their Forte names, together with their degrees of transpositional and inversional symmetries, and some other details. For every set class, its complement is easily found, as well as its Z-related set class, if it has one. The prime form chosen for characterizing the set classes is the IF, since it has proved to be more versatile than the others and, additionally, some characteristics of the table are based on it. To simplify the notation, the prime IFs are written without braces. Moreover, when a prime IF contains several semitones in a row they are represented as a power of 1 (for example, 111 is represented as 1^3). Table 1 has been developed in the following way:

1) Set classes are first grouped by cardinality and each of these groups is called a *period*. Periods are arranged by increasing cardinality and are represented by the cardinality followed by a hyphen (left column in Table 1), as in the initial part of Forte names. To make the table more compact, periods 0, 1, and 2 are placed in a single row, as are periods 12, 11, and 10.
2) Each period starts with the set class whose pitch classes are the closest together (that is, in a chromatic sequence) and ends with the set class whose pitch classes are the most evenly spaced. Thus, for example, period 4 starts with {1119} and ends with {3333}.
3) Within a period, set classes are arranged by decreasing ICV and are assigned the same ordinals as in Forte names (the big numbers in the cells of Table 1). This way, the number of smaller intervals in the set classes decreases within a period, which matches the previous criterion; and, additionally, each set class and its complement are placed in the same column. Z-related set classes share a cell, so that they are easily identified and each cell has a unique ICV (for $c \geq 2$). For each Z-related pair, the member with greater TCV (hard) is placed in the upper part of the cell. Remember that the complement of a hard set class is a soft one and vice versa. For example, the complement of 5-Z12 (soft) is 7-Z12 (hard). And, since 5-36 is not inversionally symmetrical, the complements of 5-Z36a and 5-Z36b (hard) are, respectively, 7-Z36b and 7-Z36a (soft).
4) Set classes being inversionally symmetrical ($s_I = s_T$) have the ordinal underlined. Regarding the degree of transpositional symmetry, when $s_T > 1$ it is given as a superscript on the ordinal (no superscript means $s_T = 1$).
5) For set classes not being inversionally symmetrical ($s_I = 0$), those whose prime IFs are type b have an asterisk on the ordinal, and those whose Rahn (and Forte) prime forms are type b have a superscript with a plus sign (+). Otherwise, they are type a. In the special case of 6-14, a superscript with the equal sign (=) is used to indicate that its two types, 6-14a and 6-14b, are the only ones with the same TTV and the type a corresponds to the prime IF (but the Rahn and Forte prime forms are type b, so a plus sign is included, too). Additionally, both are self-complementary.
6) The prime IFs are given just below the ordinals. Furthermore, in order to facilitate finding a prime IF in its corresponding period, set classes with the same number of semitones (number of 1’s in the prime IF or first element in the ICV) are assigned

the same cell colour (white or grey in the printed version of the journal). And that colour is also assigned to their complements.

To sum up, the periodic table provides the following information: the Forte names, the degrees of transpositional and inversional symmetries, the Z and complement relations, the prime IFs and the types of all prime forms. Additionally, Forte and Rahn normal forms are easily obtained from the prime IFs and their corresponding types. Moreover, given the prime IF of a set class, it is easy to find it in the table.

Table 2 shows, for each period, the number of set classes having the same number of semitones, that is, with the same cell colour in Table 1 (a similar arrangement is performed in both tables). As well, it shows, for each period, the number of set classes, set types and pitch-class sets. The special case of set class 0-1 (null set) is also included, so the total number of pitch-class sets is, of course, $2^{12} = 4096$. Note the symmetries between complementary periods.

Table 1. Periodic table of set classes.

PERIODIC TABLE OF SET CLASSES

0-	1^{12} –		1-	1 0		2-	1 1B	2 2A	3 39	4 48	5 57	6^{2} 66						
3-	1 1^{2}A	2 129	3 138	4 147	5 156	6 228	7 237	8 246	9 255	10 336	11 345	12^{3} 444						
4-	1 $1^{3}9$	2 $1^{2}28$	3 1218	4 $1^{2}37$	5 $1^{2}46$	6 $1^{2}55$	7 1317	8 1416	9^{2} 1515	10 1272	11 1227	12^{+} 1263	13 1236	14^{+} 1254	29 1245			
	16 1425	17 1353	18 1335	19 1344	20 1434	21 2226	22 2235	23 2325	24 2244	25^{2} 2424	26 2343	27 2334	28^{4} 3333		15 1326			
5-	1 $1^{4}8$	2 $1^{3}27$	3 $1^{2}217$	4 $1^{3}36$	5 $1^{3}45$	6 $1^{2}316$	7 $1^{2}415$	8 $1^{2}262$	9 $1^{2}226$	10 12126	11^{+} $1^{2}253$	36 $1^{2}235$ 12 12216	13 $1^{2}244$	14 $1^{2}325$	15 $1^{2}424$	16 12135	37 $1^{2}343$ 17 12144	38 $1^{2}334$ 18 12513
	19 12315	20 12414	21 13134	22 13314	23 12252	24 12225	25 12342	26^{+} 12243	27 12234	28^{+} 12423	29 12324	30 13224	31 12333	32 13233	33 22224	34 22233	35 22323	
6-	1 $1^{5}7$	2 $1^{4}26$	36 $1^{4}35$ 3 $1^{3}216$	37 $1^{4}44$ 4 $1^{2}21^{2}6$	5 $1^{3}315$	38 $1^{3}414$ 6 $1^{2}31^{2}5$	7^{2} $1^{2}41^{2}4$	8 $1^{3}252$	9 $1^{3}225$	39^{+} $1^{3}243$ 10* $1^{2}2152$	40 $1^{3}234$ 11 $1^{2}2125$	41 $1^{3}324$ 12 $1^{2}2215$	42 $1^{3}333$ 13 121215	$14^{=+}$ $1^{2}2143$	15 $1^{2}2134$	16 $1^{2}2413$	43 $1^{2}3124$ 17 $1^{2}2314$	18^{*+} $1^{2}3214$
	44 $1^{2}3133$ 19 121314	20^{3} 131313	21 $1^{2}2242$	22 $1^{2}2224$	45 $1^{2}2332$ 23 121242	46 $1^{2}2233$ 24 121224	47 $1^{2}2323$ 25 122124	48 $1^{2}3223$ 26 122214	27 121233	49 121323 28 122133	29 123213 50 123132	30^{2} 123123	31 122313	32 122322	33 122232	34 122223	35^{6} 222222	
7-	1 $1^{6}6$	2 $1^{5}25$	3 $1^{5}34$	4 $1^{4}215$	5 $1^{3}21^{2}5$	6 $1^{4}314$	7 $1^{3}31^{2}4$	8 $1^{4}242$	9 $1^{4}224$	10 $1^{4}233$	11* $1^{3}2142$	12 $1^{4}323$ 36 $1^{3}2124$	13 $1^{2}21^{2}24$	14 $1^{3}2214$	15 $1^{2}221^{2}4$	16 $1^{3}2133$	17 $1^{2}21^{2}33$ 37 $1^{2}21412$	18 $1^{3}2313$ 38 $1^{2}21214$
	19 $1^{3}3123$	20 $1^{2}231^{2}3$	21 $1^{2}21313$	22 $1^{2}31213$	23 $1^{3}2232$	24 $1^{3}2223$	25 $1^{2}21232$	26* $1^{2}21322$	27 $1^{2}21223$	28* $1^{2}22132$	29 $1^{2}22123$	30 $1^{2}22213$	31 1212123	32 1212213	33 $1^{2}22222$	34 1212222	35 1221222	
8-	1 $1^{7}5$	2 $1^{6}24$	3 $1^{6}33$	4 $1^{5}214$	5 $1^{4}21^{2}4$	6 $1^{3}21^{3}4$	7 $1^{5}313$	8 $1^{4}31^{2}3$	9^{2} $1^{3}31^{3}3$	10 $1^{5}232$	11 $1^{5}223$	12* $1^{4}2132$	13 $1^{4}2123$	14* $1^{3}21^{2}32$	15 $1^{4}2213$			
	16 $1^{3}221^{2}3$	17 $1^{3}21312$	18 $1^{3}21213$	19 $1^{2}21^{2}213$	20 $1^{2}2121^{2}3$	21 $1^{4}2222$	22 $1^{3}21222$	23 $1^{3}22122$	24 $1^{2}21^{2}222$	25^{2} $1^{2}221^{2}22$	26 $1^{2}212212$	27 $1^{2}212122$	28^{4} 12121212		29 $1^{3}21^{2}23$			
9-	1 $1^{8}4$	2 $1^{7}23$	3 $1^{6}213$	4 $1^{5}21^{2}3$	5 $1^{4}21^{3}3$	6 $1^{6}222$	7 $1^{5}2122$	8 $1^{4}21^{2}22$	9 $1^{3}21^{3}22$	10 $1^{4}21212$	11 $1^{3}21^{2}212$	12^{3} $1^{2}21^{2}21^{2}2$						
12-	1^{12} 1^{12}		11-	1 $1^{10}2$		10-	1 $1^{9}3$	2 $1^{8}22$	3 $1^{7}212$	4 $1^{6}21^{2}2$	5 $1^{5}21^{3}2$	6^{2} $1^{4}21^{4}2$						

Table 2. Number of set classes, set types and pitch-class sets.

Period	Set Classes with the same Number of Semitones													Set Classes	Set Types	Pitch-Class Sets
	12	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**0**			
0-													1	1	1	1
1-													1	1	1	12
2-												1	5	6	6	66
3-											1	4	7	12	19	220
4-										1	8	12	8	29	43	495
5-									1	6	18	10	3	38	66	792
6-								1	9	19	17	3	1	50	80	924
7-							1	6	18	10	3			38	66	792
8-						1	8	12	8					29	43	495
9-					1	4	7							12	19	220
10-				1	5									6	6	66
11-			1											1	1	12
12-	1													1	1	1
Total	1	0	1	1	6	5	16	19	36	36	47	30	26	224	352	4096

7. Conclusions

In this article, I represent each set type by the intervallic form, while the trichord-type vector fully characterizes its sonority, with only one exception. As well, I derive the formulas relating the trichord-type vectors of a set type and its complement. When dealing with set classes instead of set types, we can use the trichord-class vector, which is easily obtained from the trichord-type vector and, unlike the interval-class vector, fully characterizes a set class. Then, I provide a detailed list of set classes and types, including for each one the normal intervallic form, both Rahn and Forte normal forms, an extended Forte name, the interval-class vector and the trichord-type vector. The inclusion of this last characteristic represents a relevant contribution compared to the usual lists found in the literature. Finally, I develop a compact periodic table containing all set classes, which shows their main characteristics and relationships at a glance. The interval-class, trichord-class, and trichord-type vectors played a crucial role in arranging the set classes and types, both in the detailed list and the periodic table.

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Appendix 1: Formulas for the Interval-Class and Trichord-Type Vectors

This Appendix gives the derivation of the formulas included in sections 3 and 4 relating the ICVs and TTVs of a set class or set type and its complement. Now, another representation of a pitch-class set, considered as a subset of $\mathbb{Z}_{12}$, will be used: the *characteristic function*, which will be written in brackets without commas. For example, the pitch-class set [95A24], considered in section 2, has the characteristic function [001011000110], as it contains the pitch classes 2, 4, 5, 9 and A. The characteristic function of its complement is obtained by simply substituting every 1 with a 0 and vice versa, that is, [110100111001]. When dealing with a set class or set type, the characteristic function of any of their pitch-class sets can be used. Lewin (1960) provides the first formula in this respect, corresponding to the so-called "intervallic content," which is similar to the ICV. The ICV formula can be derived as follows:

Let S and S' be the characteristic functions of a given set class and its complement. Their circular autocorrelations modulo 12 are, respectively,

$$P(n) = \sum_{k=0}^{11} S(k)S(k+n)\,, \qquad P'(n) = \sum_{k=0}^{11} S'(k)S'(k+n)\,, \qquad n = 0, \dots, 11, \tag{A1}$$

where $k+n$ is understood modulo 12. Note that $P(n) = P(12-n)$.

As $S'(k) = 1 - S(k)$,

$$P'(n) = \sum_{k=0}^{11} [1 - S(k)][1 - S(k+n)] = 12 - 2c + P(n), \tag{A2}$$

c being the cardinality of the set class S.

The ICV $[d_1 \cdots d_6]$ of S is then

$$d_n = \frac{P(n)}{s_n}, \quad n = 1, \dots, 6, \tag{A3}$$

where s_n is the degree of transpositional symmetry (written as s_T in section 2) of the n-th dyad, which is $s_n = 1$ for $n = 1, \dots, 5$ and $s_6 = 2$, the latter corresponding to the tritone. Due to its symmetry, each tritone in S adds 2 to $P(6)$, so it is necessary to divide it by $s_6 = 2$ to obtain the correct value of d_6. Thus, formulas (A2) and (A3) give the vector equation

$$\begin{bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ d'_4 \\ d'_5 \\ d'_6 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} + (6-c) \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}. \tag{A4}$$

Therefore, given the ICV of a set class, equation (A4) gives the ICV of its complement.

For the TTV formula, I generalize the previous procedure by defining the functions

$$\begin{aligned} Q(i,j) &= \sum_{k=0}^{11} S(k)S(k+i)S(k+j) \\ Q'(i,j) &= \sum_{k=0}^{11} S'(k)S'(k+i)S'(k+j) \end{aligned} \qquad i,j = 0,\ldots,11, \tag{A5}$$

where $k + i$ and $k + j$ are understood modulo 12.

As $S'(k) = 1 - S(k)$,

$$Q'(i,j) = \sum_{k=0}^{11} [1 - S(k)][1 - S(k+i)][1 - S(k+j)], \tag{A6}$$

which, taking into account that

$$\sum_{k=0}^{11} S(k+i)S(k+j) = P(j-i), \tag{A7}$$

gives

$$Q'(i,j) = 12 - 3c + P(i) + P(j) + P(j-i) - Q(i,j), \tag{A8}$$

c being the cardinality of the set type S. To calculate the elements of the TTV, the functions $Q(i,j)$ and $Q'(i,j)$ are written as $Q(n)$ and $Q'(n)$ following the relations in Table A.1 (obtained from the Rahn or Forte normal forms of the 19 trichord types, excluding the first 0).

Table A.1. Relation between n and the pair i, j for $n = 1, \ldots, 19$

n		1	2	3	4	5	6	7	8	9
i, j		1, 2	1, 3	2, 3	1, 4	3, 4	1, 5	4, 5	1, 6	5, 6
n	10	11	12	13	14	15	16	17	18	19
i, j	2, 4	2, 5	3, 5	2, 6	4, 6	2, 7	3, 6	3, 7	4, 7	4, 8

The TTV $[t_1 \cdots t_{19}]$ of S is then

$$t_n = \frac{Q(n)}{r_n}, \quad n = 1, \dots, 19, \tag{A9}$$

where r_n is the degree of transpositional symmetry (written as s_T in section 2) of the n-th trichord type, which is $r_n = 1$ for $n = 1, \dots, 18$ and $r_{19} = 3$, the latter corresponding to the augmented triad. Due to its symmetry, each augmented triad in S adds 3 to $Q(19)$, so it is necessary to divide it by $r_{19} = 3$ to obtain the correct value of t_{19}.

Now, let T be the matrix whose rows are the ICVs of the 19 trichord types, which is

$$\mathrm{T} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}. \tag{A10}$$

Then, each trichord type n contains 3 dyads that are given by the corresponding row of matrix T. And, taking into account the relation between n and the pair i, j in Table A.1, the functions $P(i)$, $P(j)$, and $P(j-i)$ in (A8) are precisely the number of times each of those dyads is contained in S (except the tritones, which are recorded twice). On the other hand, the actual dyads in S are given by its ICV (d_k, $k = 1, \dots, 6$). Therefore, for each trichord type n,

$$P(i) + P(j) + P(j-i) = \sum_{k=1}^{6} T_{nk} d_k s_k, \qquad n = 1, \dots, 19 \tag{A11}$$

Thus, formulas (A8), (A9), and (A11) give the matrix equation

$$\begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ \vdots \\ t'_{18} \\ t'_{19} \end{bmatrix} = \mathrm{T_h} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix} + (4-c) \begin{bmatrix} 3 \\ 3 \\ 3 \\ \vdots \\ 3 \\ 1 \end{bmatrix}. \tag{A12}$$

$\mathrm{T_h}$ being the matrix T, but by multiplying its last column by $s_6 = 2$ and dividing its last row by $r_{19} = 3$, that is,

$$\mathrm{T_h} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{A13}$$

Therefore, given the TTV and the ICV of a set type, equation (A12) gives the TTV of its complement.

Aditionally, it is possible to obtain the ICV of a set type S from its TTV. Each column k in matrix T gives the number of times the corresponding dyad is contained in the different trichord types. And the actual trichord types in S are given by its TTV (t_n, $n = 1, \dots, 19$). On the other hand, if the cardinality of S is $c \geq 3$, each dyad in S forms a trichord type with each of the other pitch classes in S, so that dyad is contained a total of $c - 2$ times in the trichord types in S. Therefore, for each dyad k,

$$\sum_{n=1}^{19} T_{nk} t_n = (c-2) d_k, \qquad k = 1, \dots, 6, \qquad c \geq 3, \tag{A14}$$

which can be written in matrix form as

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \frac{1}{c-2}\mathrm{T}' \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix}, \qquad c \geq 3, \tag{A15}$$

T' being the transpose of matrix T in (A10).

Thus, from (A12) and (A15),

$$\begin{bmatrix} t'_1 \\ t'_2 \\ t'_3 \\ \vdots \\ t'_{18} \\ t'_{19} \end{bmatrix} = \left(\frac{1}{c-2}\mathrm{T_h T'} - \mathrm{I}\right) \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{18} \\ t_{19} \end{bmatrix} + (4-c) \begin{bmatrix} 3 \\ 3 \\ 3 \\ \vdots \\ 3 \\ 1 \end{bmatrix}, \qquad c \geq 3, \tag{A16}$$

where I is the identity matrix of size 19, and the matrix product $\mathrm{T_h T'}$ is

$$\mathrm{T_h T'} = \begin{bmatrix}
5 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 0 \\
3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 0 \\
2 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 3 \\
2 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & 3 \\
2 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & 3 \\
2 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 4 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 4 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 \\
2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 5 & 2 & 2 & 3 & 3 & 2 & 0 & 1 & 1 & 3 \\
1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 1 & 1 & 3 & 2 & 2 & 2 & 0 \\
1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 1 & 1 & 3 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 1 & 1 & 4 & 4 & 1 & 2 & 1 & 1 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 1 & 1 & 4 & 4 & 1 & 2 & 1 & 1 & 3 \\
1 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 1 & 1 & 5 & 0 & 2 & 2 & 0 \\
0 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 0 & 6 & 2 & 2 & 0 \\
0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 3
\end{bmatrix}. \tag{A17}$$

Therefore, given the TTV of a set type, equation (A16) gives the TTV of its complement.

Appendix 2: List of Set Classes and Types

Columns: 1)General ordinal – 2)Normal IF – 3)Rahn normal form – 4)Extended Forte name – 5)ICV – 6)TTV

c = 0

0-[12]	-	-	0-1	000000	000000000-0000000000

c = 1

1-	0	0	1-1	000000	000000000-0000000000

c = 2

2-	1B	01	2-1	100000	000000000-0000000000
3-	2A	02	2-2	010000	000000000-0000000000
4-	39	03	2-3	001000	000000000-0000000000
5-	48	04	2-4	000100	000000000-0000000000
6-	57	05	2-5	000010	000000000-0000000000
7-[2]	66	06	2-6	000001	000000000-0000000000

c = 3

8-	11A	012	3-1	210000	100000000-0000000000
9	129	013	3-2a	111000	010000000-0000000000
10	192	023	3-2b	111000	001000000-0000000000
11	138	014	3-3a	101100	000100000-0000000000
12	183	034	3-3b	101100	000010000-0000000000
13	147	015	3-4a	100110	000001000-0000000000
14	174	045	3-4b	100110	000000100-0000000000
15	156	016	3-5a	100011	000000010-0000000000
16	165	056	3-5b	100011	000000001-0000000000
17-	228	024	3-6	020100	000000000-1000000000
18	237	025	3-7a	011010	000000000-0100000000
19	273	035	3-7b	011010	000000000-0010000000
20	246	026	3-8a	010101	000000000-0001000000
21	264	046	3-8b	010101	000000000-0000100000
22-	255	027	3-9	010020	000000000-0000010000
23-	336	036	3-10	002001	000000000-0000001000
24	345	037	3-11a	001110	000000000-0000000100
25	354	047	3-11b	001110	000000000-0000000010
26-[3]	444	048	3-12	000300	000000000-0000000001

c = 4

27-	1119	0123	4-1	321000	211000000-0000000000
28	1128	0124	4-2a	221100	110100000-1000000000
29	1182	0234	4-2b	221100	101010000-1000000000
30-	1218	0134	4-3	212100	011110000-0000000000
31	1137	0125	4-4a	211110	100101000-0100000000
32	1173	0345	4-4b	211110	100010100-0010000000
33	1146	0126	4-5a	210111	100001010-0001000000
34	1164	0456	4-5b	210111	100000101-0000100000
35-	1155	0127	4-6	210021	100000011-0000010000
36-	1317	0145	4-7	201210	000111100-0000000000
37-	1416	0156	4-8	200121	000001111-0000000000

c = 12

351-[12]	111111111111	0123456789AB	12-1	CCCCC6	CCCCCCCCC-CCCCCCCCC4

c = 11

350-	11111111112	0123456789A	11-1	AAAAA5	999999999-9999999993

c = 10

344-	1111111113	0123456789	10-1	988884	877777777-6666666662
345-	1111111122	012345678A	10-2	898884	777666666-8777776662
346-	1111111212	012345679A	10-3	889884	677776666-6776668772
347-	1111112112	012345689A	10-4	888984	666777766-7667766773
348-	1111121112	012345789A	10-5	888894	666667777-6776686772
349-[2]	1111211112	012346789A	10-6	888885	666666688-6668868662

c = 9

325-	111111114	012345678	9-1	876663	766555555-5444443331
326	111111123	012345679	9-2a	777663	665554444-5554445441
327	111111132	023456789	9-2b	777663	656554444-5554445441
328	111111213	012345689	9-3a	767763	555655544-4444435552
329	111111312	013456789	9-3b	767763	555565544-4444435552
330	111112113	012345789	9-4a	766773	544556555-4444453552
331	111113112	012456789	9-4b	766773	544555655-4444453552
332	111121113	012346789	9-5a	766674	544445576-3445555441
333	111131112	012356789	9-5b	766674	544445567-3445555441
334-	111111222	01234568A	9-6	686763	555444433-7556653442
335	111112122	01234578A	9-7a	677673	455444444-5654465551
336	111112212	01345678A	9-7b	677673	455444444-5564465551
337	111121122	01234678A	9-8a	676764	444444455-6447645442
338	111122112	01245678A	9-8b	676764	444444455-6446745442
339-	111211122	01235678A	9-9	676683	444335555-5664473551
340-	111121212	01234679A	9-10	668664	355553355-3555538551
341	111211212	01235679A	9-11a	667773	344555544-4554455652
342	111212112	01245679A	9-11b	667773	344555544-4554455562
343-[3]	112112112	01245689A	9-12	666963	333666633-6336633663

c = 8

282-	11111115	01234567	8-1	765442	655443333-4332222110
283	11111124	01234568	8-2a	665542	554433322-4333322221
284	11111142	02345678	8-2b	665542	545343322-4333322221
285-	11111133	01234569	8-3	656542	544443322-3332214331
286	11111214	01234578	8-4a	655552	444434333-3322232331
287	11111412	01345678	8-4b	655552	444343433-3232232331
288	11112114	01234678	8-5a	654553	433334354-3224332221
289	11114112	01245678	8-5b	654553	433333445-3223432221
290-	11121114	01235678	8-6	654463	433224455-2333342220
291-	11111313	01234589	8-7	645652	433444433-2222222442
292-	11113113	01234789	8-8	644563	422334455-1223342331

38-[2]	1515	0167	4-9	200022	000000022-0000000000
39-	1272	0235	4-10	122010	011000000-0110000000
40	1227	0135	4-11a	121110	010001000-1010000000
41	1722	0245	4-11b	121110	001000100-1100000000
42	1263	0346+	4-12a	112101	010010000-0001101000
43	1362	0236+	4-12b	112101	001100000-0001001000
44	1236	0136	4-13a	112011	010000010-0100001000
45	1632	0356	4-13b	112011	001000001-0010001000
46	1254	0457+	4-14a	111120	010000100-0000010010
47	1452	0237+	4-14b	111120	001001000-0000010100
48	1245	0137	4-Z29a[15]	111111	010000001-0001000100
49	1542	0467	4-Z29b[15]	111111	001000010-0000100010
50	1326	0146	4-Z15a[29]	111111	000100010-0010100000
51	1623	0256	4-Z15b[29]	111111	000010001-0101000000
52	1425	0157	4-16a	110121	000001001-0000110000
53	1524	0267	4-16b	110121	000000110-0001010000
54-	1353	0347	4-17	102210	000110000-0000000110
55	1335	0147	4-18a	102111	000100001-0000001010
56	1533	0367	4-18b	102111	000010010-0000001100
57	1344	0148	4-19a	101310	000100100-0000000101
58	1443	0348	4-19b	101310	000011000-0000000011
59-	1434	0158	4-20	101220	000001100-0000000110
60-	2226	0246	4-21	030201	000000000-2001100000
61	2235	0247	4-22a	021120	000000000-1100010010
62	2253	0357	4-22b	021120	000000000-1010010100
63-	2325	0257	4-23	021030	000000000-0110020000
64-	2244	0248	4-24	020301	000000000-1001100001
65-[2]	2424	0268	4-25	020202	000000000-0002200000
66-	2343	0358	4-26	012120	000000000-0110000110
67	2334	0258	4-27a	012111	000000000-0100101100
68	2433	0368	4-27b	012111	000000000-0011001010
69-[4]	3333	0369	4-28	004002	000000000-0000004000

c = 5

70-	11118	01234	5-1	432100	322110000-1000000000
71	11127	01235	5-2a	332110	221101000-1110000000
72	11172	02345	5-2b	332110	212010100-1110000000
73	11217	01245	5-3a	322210	111211100-1100000000
74	11712	01345	5-3b	322210	111121100-1010000000
75	11136	01236	5-4a	322111	211101010-0101001000
76	11163	03456	5-4b	322111	211010101-0010101000
77	11145	01237	5-5a	321121	211001011-0001010100
78	11154	04567	5-5b	321121	211000111-0000110010
79	11316	01256	5-6a	311221	100112111-0101000000
80	11613	01456	5-6b	311221	100111211-0010100000
81	11415	01267	5-7a	310132	100001132-0001010000
82	11514	01567	5-7b	310132	100001123-0000110000
83-	11262	02346	5-8	232201	111110000-2001101000
84	11226	01246	5-9a	231211	110101010-2011100000
85	11622	02456	5-9b	231211	101010101-2101100000

293-[2]	11131113	01236789	8-9	644464	422224466-0224444220
294-	11111232	02345679	8-10	566452	444332222-4442244330
295	11111223	01234579	8-11a	565552	443333222-4343342331
296	11111322	02456789	8-11b	565552	434332322-4433342331
297	11112312*	01345679	8-12a	556543	343342233-3333415331
298	11112132*	02345689	8-12b	556543	334432233-3334315331
299	11112123	01234679	8-13a	556453	343332243-2433335330
300	11113212	02356789	8-13b	556453	334332234-2343335330
301	11123112*	01245679	8-14a	555562	332333433-3442242341
302	11121132*	02345789	8-14b	555562	323334333-3442242431
303	11112213	01234689	8-Z15a[29]	555553	333323343-3233434331
304	11113122	01356789	8-Z15b[29]	555553	333233334-3324334331
305	11121123	01235679	8-Z29a[15]	555553	332333334-3334334321
306	11132112	02346789	8-Z29b[15]	555553	323333343-3333434231
307	11122113	01235789	8-16a	554563	322224345-3333442331
308	11131122	01246789	8-16b	554563	322223454-3334342331
309-	11121312	01345689	8-17	546652	233444422-2332224442
310	11121213	01235689	8-18a	546553	233433334-1333325341
311	11131212	01346789	8-18b	546553	233343343-1333325431
312	11211213	01245689	8-19a	545752	222544522-3223322542
313	11211312	01345789	8-19b	545752	222455422-3223322452
314-	11212113	01245789	8-20	545662	222444433-2332242442
315-	11112222	0123468A	8-21	474643	333222222-6336632222
316	11121222	0123568A	8-22a	465562	233223322-4543352341
317	11122212	0134568A[l]	8-22b	465562	233223322-4453352431
318-	11122122	0123578A	8-23	465472	233113333-4552262440
319-	11211222	0124568A	8-24	464743	222333322-6226622332
320-[2]	11221122	0124678A	8-25	464644	222222244-6226624222
321-	11212212	0134578A[m]	8-26	456562	133333322-3442254441
322	11212122	0124578A	8-27a	456553	133332233-3433435431
323	11221212	0134678A[n]	8-27b	456553	133332233-3344335341
324-[4]	12121212	0134679A	8-28	448444	044440044-0444408440

c = 7

216-	1111116	0123456	7-1	654321	544332211-3221101000
217	1111125	0123457	7-2a	554331	443322111-3221121110
218	1111152	0234567	7-2b	554331	434231211-3221121110
219	1111134	0123458	7-3a	544431	433322211-2211111221
220	1111143	0345678	7-3b	544431	433232211-2121111221
221	1111215	0123467	7-4a	544332	333322132-2212112110
222	1111512	0134567	7-4b	544332	333231223-2121212110
223	1112115	0123567	7-5a	543342	322223233-2222121100
224	1115112	0124567	7-5b	543342	322222333-2221221010
225	1111314	0123478	7-6a	533442	322223233-1102121221
226	1111413	0145678	7-6b	533442	322222333-1011221221
227	1113114	0123678	7-7a	532353	311113354-0113231110
228	1114113	0125678	7-7b	532353	311113345-0112331110
229-	1111242	0234568	7-8	454422	333221111-3223302111
230	1111224	0123468	7-9a	453432	332212121-3123321111
231	1111422	0245678	7-9b	453432	323121212-3213321111

86	12126	01346	5-10a	223111	021110010-0110101000
87	12162	02356	5-10b	223111	012110001-0111001000
88	11253	03457[+]	5-11a	222220	110110100-1010010110
89	11352	02347[+]	5-11b	222220	101111000-1100010110
90	11235	01247	5-Z36a[12]	222121	110100011-1100011010
91	11532	03567	5-Z36b[12]	222121	101010011-1010011100
92-	12216	01356	5-Z12[36]	222121	011001111-1110001000
93	11244	01248	5-13a	221311	110100101-1001100101
94	11442	02348	5-13b	221311	101011010-1001100011
95	11325	01257	5-14a	221131	100101011-0110120000
96	11523	02567	5-14b	221131	100010111-0111020000
97-	11424	01268	5-15	220222	100001111-0002210000
98	12135	01347	5-16a	213211	011210001-0001001110
99	12153	03467	5-16b	213211	011120010-0000101110
100-	11343	03458	5-Z37[17]	212320	100111100-0110000111
101-	12144	01348	5-Z17[37]	212320	011111100-0000010111
102	11334	01258	5-Z38a[18]	212221	100101101-0100101110
103	11433	03678	5-Z38b[18]	212221	100011110-0011001110
104	12513	01457	5-Z18a[38]	212221	010111101-0000111010
105	13152	02367	5-Z18b[38]	212221	001111110-0001011100
106	12315	01367	5-19a	212122	010010022-0101001100
107	13215	01467	5-19b	212122	001100022-0010101010
108	12414	01568[a]	5-20a	211231	010001211-0001010110
109	14142	02378[b]	5-20b	211231	001002111-0000110110
110	13134	01458	5-21a	202420	000211200-0000000211
111	13143	03478	5-21b	202420	000122100-0000000121
112-	13314	01478	5-22	202321	000111111-0000001111
113	12252	02357	5-23a	132130	011001000-1120020100
114	12522	02457	5-23b	132130	011000100-1210020010
115	12225	01357	5-24a	131221	010001001-2011110100
116	15222	02467	5-24b	131221	001000110-2101110010
117	12342	02358	5-25a	123121	011000010-0210101110
118	12432	03568	5-25b	123121	011000001-0121001110
119	12243	03468[+]	5-26a	122311	010011000-1011101011
120	13422	02458[+]	5-26b	122311	001100100-1101101101
121	12234	01358	5-27a	122230	010001100-1110010120
122	14322	03578	5-27b	122230	001001100-1110010210
123	12423	02568[+]	5-28a	122212	010010001-0102201100
124	13242	02368[+]	5-28b	122212	001100010-0012201010
125	12324	01368	5-29a	122131	010000110-0111021010
126	14232	02578	5-29b	122131	001001001-0110121100
127	13224	01468	5-30a	121321	000100110-1011110101
128	14223	02478	5-30b	121321	000011001-1101110011
129	12333	01369	5-31a	114112	010010010-0100104100
130	13332	02369	5-31b	114112	001100001-0011004010
131	13233	01469	5-32a	113221	000110010-0110101210
132	13323	02569[c]	5-32b	113221	000110001-0111001120
133-	22224	02468	5-33	040402	000000000-3003300001
134-	22233	02469	5-34	032221	000000000-2111111110
135-	22323	02479	5-35	032140	000000000-1220030110

232	1111233	0123469	7-10a	445332	332221121-2221214220
233	1111332	0234569	7-10b	445332	323221112-2222114220
234	1112412*	0134568	7-11a	444441	232222311-2231121221
235	1112142*	0234578	7-11b	444441	223223211-2321121221
236-	1111323	0123479	7-Z12[36]	444342	322221122-1222232220
237	1112124	0123568	7-Z36a[12]	444342	232212222-1322222120
238	1114212	0235678	7-Z36b[12]	444342	223122222-1232222210
239	1121124	0124568	7-13a	443532	221322312-3113311211
240	1121142	0234678	7-13b	443532	212233221-3113311121
241	1112214	0123578	7-14a	443352	222103233-2221231220
242	1114122	0135678	7-14b	443352	222012333-2222131220
243-	1122114	0124678	7-15	442443	211112244-3113321111
244	1112133	0123569	7-16a	435432	222322212-1222104221
245	1113312	0134569	7-16b	435432	222232221-1221204221
246-	1121133	0124569	7-Z17[37]	434541	211333311-2221111331
247-	1121412	0134578	7-Z37[17]	434541	122333311-2111121331
248	1112313	0145679[h]	7-Z18a[38]	434442	221222223-1221212231
249	1113132	0234589[i]	7-Z18b[38]	434442	212222232-1222112321
250	1121214	0124578	7-Z38a[18]	434442	122322223-1211222221
251	1141212	0134678	7-Z38b[18]	434442	122232232-1122122221
252	1113123	0123679	7-19a	434343	221122233-0213224210
253	1113213	0123689	7-19b	434343	212212233-0122324120
254	1123113	0125679[j]	7-20a	433452	210222333-1222131221
255	1131132	0234789[k]	7-20b	433452	201223233-1221231221
256	1121313	0124589	7-21a	424641	111433411-1111111432
257	1131312	0134589	7-21b	424641	111344311-1111111342
258-	1131213	0125689	7-22	424542	111333322-0112212331
259	1112232	0234579	7-23a	354351	222112111-3341141320
260	1112322	0245679	7-23b	354351	222111211-3431141230
261	1112223	0123579	7-24a	353442	221112112-3233331211
262	1113222	0246789	7-24b	353442	212111221-3323331121
263	1121232	0234679	7-25a	345342	122221121-2321234220
264	1123212	0235679	7-25b	345342	122221112-2232134220
265	1122312*	0134579	7-26a	344532	121232111-3123312231
266	1121322*	0245689	7-26b	344532	112321211-3213312321
267	1121223	0124579	7-27a	344451	121222211-2331141231
268	1132212	0245789	7-27b	344451	112222211-2331141321
269	1123122*	0135679	7-28a	344433	121121123-3213314211
270	1122132*	0234689	7-28b	344433	112211132-3123314121
271	1122123	0124679	7-29a	344352	121111232-2332132230
272	1132122	0235789	7-29b	344352	112112123-2331232320
273	1122213	0124689	7-30a	343542	111212321-3123321321
274	1131222	0135789	7-30b	343542	111123212-3213321231
275	1212123	0134679	7-31a	336333	032230032-0322305320
276	1212132	0235689	7-31b	336333	023320023-0233205230
277	1212213	0134689	7-32a	335442	022222221-1221224321
278	1213122	0135689	7-32b	335442	022222212-1222124231
279-	1122222	012468A	7-33	262623	111111111-6116611112
280-	1212222	013468A	7-34	254442	022111111-3333332221
281-	1221222	013568A	7-35	254361	022002211-3441151330

c = 6

136-	111117	012345	6-1	543210	433221100-2110000000
137	111126	012346	6-2a	443211	332211010-2111101000
138	111162	023456	6-2b	443211	323120101-2111101000
139	111135	012347	6-Z36a[3]	433221	322211011-1101011110
140	111153	034567	6-Z36b[3]	433221	322120111-1010111110
141	111216	012356	6-Z3a[36]	433221	222212111-1211001000
142	111612	013456	6-Z3b[36]	433221	222121211-1120101000
143-	111144	012348	6-Z37[4]	432321	322111111-1001110111
144-	112116	012456	6-Z4[37]	432321	211222211-2111100000
145	111315	012367	6-5a	422232	211112132-0102011100
146	111513	014567	6-5b	422232	211111223-0010211010
147-	111414	012378	6-Z38[6]	421242	211002233-0001120110
148-	113115	012567	6-Z6[38]	421242	200112233-0111120000
149-[2]	114114	012678	6-7	420243	200002244-0002220000
150-	111252	023457	6-8	343230	222111100-2220020110
151	111225	012357	6-9a	342231	221102011-2121120100
152	111522	024567	6-9b	342231	212010211-2211120010
153	111243	034568[+]	6-Z39a[10]	333321	221111101-1121101111
154	111342	023458[+]	6-Z39b[10]	333321	212111110-1211101111
155	112512*	013457	6-Z10a[39]	333321	121221101-2011111110
156	112152*	023467	6-Z10b[39]	333321	112221110-2101111110
157	111234	012358	6-Z40a[11]	333231	221101111-1210111120
158	111432	035678	6-Z40b[11]	333231	212011111-1121011210
159	112125	012457	6-Z11a[40]	333231	121211111-1210121010
160	115212	023567	6-Z11b[40]	333231	112121111-1121021100
161	111324	012368	6-Z41a[12]	332232	211101121-0112221010
162	111423	025678	6-Z41b[12]	332232	211011112-0112221100
163	112215	012467	6-Z12a[41]	332232	111101132-2111111010
164	115122	013567	6-Z12b[41]	332232	111011123-2111111100
165-	111333	012369	6-Z42[13]	324222	211111111-0111104110
166-	121215	013467	6-Z13[42]	324222	022220022-0111102110
167	112143	034578[+]	6-14a[=]	323430	111222200-1110010221
168	113412	013458[+]	6-14b[=]	323430	111222200-1110010221
169	112134	012458	6-15a	323421	111311201-1101101211
170	114312	034678	6-15b	323421	111132110-1011101121
171	112413	014568	6-16a	322431	110211311-1011110211
172	113142	023478	6-16b	322431	101123111-1101110121
173	113124	012568	6-Z43a[17]	322332	110112212-0102211110
174	114213	023678	6-Z43b[17]	322332	101112221-0012211110
175	112314	012478	6-Z17a[43]	322332	110111123-1101111111
176	114132	014678	6-Z17b[43]	322332	101111132-1011111111
177	114123*	013678[+]	6-18a	322242	110011232-0112021110
178	113214*	012578[+]	6-18b	322242	101102123-0110221110
179	113133	012569	6-Z44a[19]	313431	100222211-0111001221
180	113313	014569[d]	6-Z44b[19]	313431	100222211-0110101221
181	121314	013478	6-Z19a[44]	313431	011222211-0001011221
182	121413	014578	6-Z19b[44]	313431	011222211-0000111221
183-[3]	131313	014589	6-20	303630	000333300-0000000332
184	112242	023468	6-21a	242412	111111010-3013301011
185	112422	024568	6-21b	242412	111110101-3103301101
186	112224	012468	6-22a	241422	110101111-3013310101
187	114222	024678	6-22b	241422	101011111-3103310011
188-	112332	023469	6-Z45[23]	234222	111110011-2111114110
189-	121242	023568	6-Z23[45]	234222	022110011-0222202110
190	112233	012469	6-Z46a[24]	233331	110111110-2121111220
191	113322	024569	6-Z46b[24]	233331	101111101-2211111220
192	121224	013468	6-Z24a[46]	233331	021111110-1121121111
193	121422	024578	6-Z24b[46]	233331	012111101-1211121111
194	112323	012479	6-Z47a[25]	233241	110110111-1221031120
195	113232	023479	6-Z47b[25]	233241	101111011-1220131210
196	122124	013568	6-Z25a[47]	233241	021001211-1221021120
197	122142	023578	6-Z25b[47]	233241	012002111-1220121210
198-	113223	012579	6-Z48[26]	232341	100111111-1221130111
199-	122214	013578	6-Z26[48]	232341	011002211-2111120220
200	121233	013469	6-27a	225222	021120020-0210204210
201	121332	023569	6-27b	225222	012210002-0122004120
202-	121323	013479	6-Z49[28]	224322	011220011-0112202220
203-	122133	013569	6-Z28[49]	224322	011111111-1111104111
204-	123213	023679[e]	6-Z29[50]	224232	011111111-0111124110
205-	123132	014679	6-Z50[29]	224232	011110022-0221102220
206[2]	123123	013679	6-30a	224223	020020022-0202204200
207[2]	132132	023689	6-30b	224223	002200022-0022204020
208	122313	014579[f]	6-31a	223431	010122101-1111111131
209	131322	024589[g]	6-31b	223431	001211210-1111111311
210-	122322	024579	6-32	143250	011001100-2330040220
211	122232	023579	6-33a	143241	011001001-2231131210
212	123222	024679	6-33b	143241	011000110-2321131120
213	122223	013579	6-34a	142422	010011001-3113311111
214	132222	024689	6-34b	142422	001100110-3113311111
215-[6]	222222	02468A	6-35	060603	000000000-6006600002

Forte normal forms that are different from Rahn's:

[a] 5-20a 01378
[b] 5-20b 01578
[c] 5-32b 01479
[d] 6-Z44b[19] 012589
[e] 6-Z29[50] 013689
[f] 6-31a 013589
[g] 6-31b 014689
[h] 7-Z18a[38] 0123589
[i] 7-Z18b[38] 0146789
[j] 7-20a 0124789
[k] 7-20b 0125789
[l] 8-22b 0123579A
[m] 8-26 0124579A
[n] 8-27b 0124679A